

Bernstein-Schnabl operators: new achievements and perspectives

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Bernstein-Schnabl operators were first introduced by R. Schnabl in 1968 in the context of the sets of all probability Radon measures on compact Hausdorff spaces, in order to present a unitary treatment of the saturation problem for Bernstein operators on the unit interval and on the finite dimensional simplices and hypercubes.

These operators were referred to as Bernstein-Schnabl operators in 1971 by G. Felbecker and W. Schempp who extended their definition by considering arbitrary infinite lower triangular stochastic matrices. Finally, in 1974 M. W. Grossman systematized the same definition in the more appropriate setting of (not necessarily finite dimensional) convex compact subsets.

In 1989 the present author undertook a detailed study of these operators in the special case where they are linked to a positive (linear) projection, showing a deep interplay between them, the theory of Feller semigroups and the relevant initial-boundary value differential problems and their probabilistic counterparts. The main aspects of this theory, which has been also accomplished by other authors, is documented in Chapter 6 of [1].

Motivated by the need to enlarge the classes of both approximation and differential problems where a similar theory could be applied, recently a joint project with M. Cappelletti Montano, V. Leonessa and I. Raşa has been undertaken in order to extend the above mentioned theory by considering arbitrary positive (linear) operators instead of positive projections. This new challenging approach has disclosed new and more difficult problems.

The talk will be centered about some results obtained with this new approach whose details will appear in the forthcoming monograph [2].

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On linear shape and/or test-functions preserving operators

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We present old and new results concerning the approximation of real-valued continuous functions by means of sequences of linear operators. On one hand, we consider some modifications of classical operators which hold fixed certain polynomials, or more in general, φ -polynomials, and on the other hand, we work under the setting of the so-called simultaneous approximation assuming that the operators preserve the sign of certain derivatives of the functions. Both situations are developed by studying the classical topics within the well-known Korovkin-type approximation theory.

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On L_p multiple orthogonal polynomials

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Denote by P_n the space of real algebraic polynomials of degree at most $n-1$ and consider a multi index $\mathbf{n} := (n_1, \dots, n_d) \in \mathbb{N}^d, d \geq 1$ of length $|\mathbf{n}| := n_1 + \dots + n_d$. Then given the nonnegative weight functions $w_j \in L_\infty[a, b], 1 \leq j \leq d$ the polynomial $Q \in P_{|\mathbf{n}|+1} \setminus \{0\}$ is called a multiple orthogonal polynomial relative to \mathbf{n} and the weights $w_j, 1 \leq j \leq d$ if

$$\int_{[a,b]} w_j(x) x^k Q(x) d\mu = 0, \quad 0 \leq k \leq n_j - 1, 1 \leq j \leq d.$$

The above orthogonality relations are equivalent to the L_2 multiple best approximation conditions

$$\|Q\|_{L_2(w_j)} \leq \|Q - g\|_{L_2(w_j)}, \quad \forall g \in P_{n_j}, 1 \leq j \leq d.$$

The existence of multiple L_2 orthogonal polynomials easily follows from the solvency of the above linear system. The analogous question for multiple best L_p approximation, i.e., existence of extremal polynomial $Q_p \in P_{|\mathbf{n}|+1} \setminus \{0\}$ satisfying

$$\|Q_p\|_{L_p(w_j)} \leq \|Q_p - g\|_{L_p(w_j)}, \quad \forall g \in P_{n_j}, 1 \leq j \leq d$$

poses a more difficult non linear problem when $1 \leq p < \infty, p \neq 2$. In this paper we shall address this question and verify existence and uniqueness of multiple L_p orthogonal polynomials under proper conditions. By letting $p \rightarrow \infty$ we shall obtain certain new multiple Chebyshev polynomials.

Weighted D-T moduli revisited and applied

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We introduce weighted moduli of smoothness for functions $f \in L_p[-1, 1] \cap C^{r-1}(-1, 1)$, $r \geq 1$, that have an $(r - 1)$ st absolutely continuous derivative in $(-1, 1)$ and such that $\varphi^r f^{(r)}$ is in $L_p[-1, 1]$, where $\varphi(x) = (1 - x^2)^{1/2}$. These moduli are equivalent to certain weighted D-T moduli, but our definition is more transparent and simpler. In addition, instead of applying these weighted moduli to weighted approximation, which was the purpose of the original D-T moduli, we apply these moduli to obtain Jackson-type estimates on the approximation of functions in $L_p[-1, 1]$ (no weight), by means of algebraic polynomials. We also have some inverse theorems that yield characterization of the behavior of the derivatives of the function by means of its degree of approximation.

Further, we apply the same ideas to some direct and inverse results on weighted approximation in $L_p[-1, 1]_w$, where the weight function w is of Jacobi type.

This is a joint work with K. Kopotun (University of Manitoba, Canada) and I. A. Shevchuk (National Taras Shevchenko University of Kyiv, Ukraine).

(M, N) -Coherent Pairs of Order (m, k) and Sobolev Orthogonal Polynomials

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A pair of regular linear functionals (u, v) is said to be a (M, N) -coherent pair of order (m, k) if their corresponding sequences of monic orthogonal polynomials $\{P_n(x)\}_{n \geq 0}$ and $\{Q_n(x)\}_{n \geq 0}$ satisfy a structure relation such as

$$\sum_{i=0}^M a_{i,n} P_{n+m-i}^{(m)}(x) = \sum_{i=0}^N b_{i,n} Q_{n+k-i}^{(k)}(x), \quad n \geq 0,$$

where $a_{i,n}$ and $b_{i,n}$ are real numbers such that $a_{M,n} \neq 0$ if $n \geq M$, $b_{N,n} \neq 0$ if $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ when $i > n$.

In the first part of this talk we focus our attention in the algebraic properties of an (M, N) -coherent pair of order (m, k) . To be more precise, let us assume that $m \geq k$. If $m = k$ then u and v are related by a rational factor (in the distributional sense); if $m > k$ then u and v are semiclassical and they are again related by a rational factor.

In the second part we deal with a Sobolev type inner product defined in the linear space of polynomials with real coefficients, \mathbb{P} , as

$$\langle p(x), q(x) \rangle_\lambda = \int_{\mathbb{R}} p(x)q(x) d\mu_0(x) + \lambda \int_{\mathbb{R}} p^{(m)}(x)q^{(m)}(x) d\mu_1(x), \quad p, q \in \mathbb{P},$$

where λ is a positive real number, m is a positive integer number and (μ_0, μ_1) is a (M, N) -coherent pair of order m of positive Borel measures supported on an infinite subset of the real line, meaning that the sequences of monic orthogonal polynomials $\{P_n(x)\}_{n \geq 0}$ and $\{Q_n(x)\}_{n \geq 0}$ with respect to μ_0 and μ_1 , respectively, satisfy a structure relation as above with $k = 0$, $a_{i,n}$ and $b_{i,n}$ being real numbers fulfilling the above mentioned conditions. We generalize several recent results known in the literature in the framework of Sobolev orthogonal polynomials and their connections with coherent pairs (introduced in A. Iserles et al., J. Approx. Theory **65**, 151-175 (1991)) and their extensions. We show how to compute the coefficients of the Fourier expansion of functions on an appropriate Sobolev space (defined by the above inner product) in terms of the sequence of Sobolev orthogonal polynomials $\{S_n(x; \lambda)\}_{n \geq 0}$. A comparison between the rate of convergence of the corresponding Fourier expansions for Sobolev and standard orthogonal polynomials is done.

This is a joint work with M. N. de Jesus (School of Technology and Management, Viseu, Portugal), J. C. Petronilho (University of Coimbra, Portugal) and N. Pinzón (“Carlos III” University of Madrid, Spain).

Numerical Method for Fredholm Integral Equations with Periodic Solutions

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In this lecture we consider Fredholm integral equations of the second kind

$$f(x) = g(x) + \int_{\mathbb{R}} K(x, t) f(t) dt,$$

with a degenerate kernel

$$K(x, t) = \sum_{\nu=1}^m \gamma_{\nu}(x) \omega_{\nu}(t),$$

where ω_{ν} , $\nu = 1, \dots, m$, are different rational functions of the form

$$t \mapsto \frac{1}{(t^2 + b^2)^s}, \quad b > 0, \quad s \in \mathbb{N},$$

and the functions g and γ_{ν} , $\nu = 1, \dots, m$, are (2π) -periodic.

We present an efficient numerical method for solving this kind of Fredholm integral equations, using a transformation method for the weighted integration of periodic functions on the real line given by Mastroianni and Milovanović [1] (see also [pp. 350–361, 2]). Such integral equations can be reduced to an equivalent system of linear algebraic equations and treated by the methods of linear algebra. Numerical examples are included.

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Exact non-reflecting boundary conditions for (exterior) wave equation problems

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We consider some wave propagation problems defined on unbounded space domains, possibly having far field sources. These are defined by the wave equation in the frequency domain, in the case of harmonic waves, and in the time domain in the more general case. Although we will limit our description to the 2D case, the proposed approaches can be easily extended to 3D problems.

Since one has to compute the solution of the given PDE problem only in a bounded area surrounding a physical (bounded) domain, this area is then defined by introducing an artificial outer boundary B , where a non reflecting (or transparent) condition is imposed. The problem physical domain can even be a multi-domain, defined by the union of several disjoint domains. These domains can be convex or non-convex. To efficiently deal with these general situations, the shape of the chosen artificial boundary should be the most appropriate one for the given problem. It could be a non-convex closed curve or even the union of (disjoint) closed curves. It should not necessarily include the problem datum supports, in particular when these are away from the computational domain. Thus an appropriate boundary condition on B should be non reflecting for both outgoing and incoming waves.

The most commonly used transparent conditions are of local type, hence approximate, to reduce their computational cost. However, they hold only for convex artificial boundaries, having particular shapes; moreover, they do not satisfy most of the above requirements.

As already proposed since many years for some elliptic problems, and in particular for the Helmholtz one, non reflecting boundary conditions can be defined by proper boundary integral equations, defining a relationship between the solution of the differential problem and its normal derivative on B . Such conditions are exact, hence non local, but their computational cost is in general higher than that of the local ones. However, they have all the good properties mentioned above.

To solve the PDE problem, now defined in the chosen bounded region, we apply a finite element or finite difference scheme, to discretize the (space) computational region, and a classical (explicit or implicit) time integrator in the case of the wave equation. The associated boundary integral equation is discretized, with respect to the space variable, by using a classical collocation method. In the case of the time-domain wave equation, since the associated integral equation is of space-time type, we also need to discretize its time integral. This is performed by means of a convolution quadrature.

We will discuss all the computational issues that make the proposed approaches efficient, compare these with those of local type, and present several numerical examples which show the complete transparency of the proposed boundary conditions, in all possible situations.

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Shape preserving linear polynomial operators and their eigenvalues

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In this talk we consider linear polynomial operators and different shape properties (positivity, monotonicity, convexity, ...). We will study different situations and the estimates for their eigenvalues. Optimality of the bounds is considered.

Emerging problems in Approximation Theory for the numerical solution of nonlinear PDEs of integrable type

S. Seatzu, *University of Cagliari, Italy*

In this talk we present some open problems pertaining to the approximation theory involved in the solution of an important class of Nonlinear Partial Differential Equations (NPDEs). A general theory to solve the class of NPDEs does not seem to exist. However, for the important class of NPDEs of integrable type, a path to solve the corresponding Initial Value Problem (IVP) is given by the Inverse Scattering Transform (IST) technique. After a brief survey of the main steps required by the new formulation of this method, recently proposed by C. van der Mee, we present two problems in Approximation Theory whose solution greatly influences the overall effectiveness of the method. In fact, the effectiveness of the method depends on the solution of some Volterra systems with structured kernels on unbounded domains and the approximation of data by using a sum of exponentials. After an illustration of the method developed and the results of our numerical experiments, we will highlight the difficulties that remain to overcome in order to have a completely reliable numerical method.

This work is joint with L. Fermo and C. van der Mee (University of Cagliari, Italy).

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A weighted generalization of Szász-Mirakyan and Butzer operators

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An operator is introduced for the weighted L^p approximation on the semi-axis with weight

$$w_\beta(x) = \begin{cases} x^\alpha \exp(-x^\beta) & \text{if } 0 < \beta < 1 \\ x^\alpha \exp(-cx \log x) & \text{if } \beta = 1 \end{cases}, \quad \alpha > 1/p,$$

allowing a much wider class of functions than in the case of classical Szász-Mirakyan-Kantorovich operators. It is shown that in case $1/2 < \beta \leq 1$, only pointwise convergence holds, while for $0 < \beta \leq 1/2$, direct and converse theorems in terms of a weighted modulus of smoothness hold. A Voronovskaya-type relation is also proved.

This work is joint with B. Della Vecchia (“La Sapienza” University of Rome, Italy) and G. Mastroianni (University of Basilicata, Italy).

The polynomial inverse image method

V. Totik, *University of Szeged, Hungary and University of South Florida, USA*

We review a method, based on taking polynomial inverse images, with which rather general theorems can be proven from their simpler special cases. There are two version:

1. Transforming results known on an interval to a general compact subset of the real line.
2. Transforming results known on the unit circle to results on unions of Jordan curves.

Concrete applications are mentioned in connection with polynomial inequalities, asymptotics of Christoffel functions and approximation on general compact sets.

Some interpolatory problems

P. Vértesi, *Alfréd Rényi Institute of Mathematics of the Hungarian Academy of Sciences, Hungary*

In this lecture we intend to mention some problems and theorems of the classical and weighted interpolation. Many new results are mentioned; some of them were proved in cooperation with Professor Giuseppe Mastroianni and his collaborators.