Generalized Adams Product Quadrature Rules for Fractional Differential Equations

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An initial value problem for a Fractional Differential Equation (FDE) can be reformulated as a Volterra integral equation of the second kind with weakly singular kernel. Its solution can be approximated by applying suitable convolution quadratures. Methods of this type currently available in the literature are, for example, Adams product quadrature rules or Fractional Linear Multistep Methods, [2, 3]. It is known that the order of an A-stable convolution quadrature cannot exceed two, [3]. Clearly, this result represents an extension of the second Dahlquist barrier for linear multistep methods (LMMs) for ordinary differential equations. This barrier has been overcomed by using LMMs as Boundary Value Methods (BVMs), namely by completing the discrete problem generated by a LMM with suitable boundary conditions, [1]. By virtue of this result, we have investigated if the BVM approach is successful in overcoming the barrier established in [3]. This led us to derive a generalized version of implicit Adams product quadrature rules called Fractional Generalized Adams Methods (FGAMs). We will present these new schemes and discuss their accuracy and stability properties. In particular, we will show that they are always A-stable in a generalized sense. Finally, the results of a numerical experiment confirming the valuable properties of this approach will be reported.

References

[1] L. Brugnano and D. Trigiante, *Solving ODEs by Linear Multistep Initial and Boundary Value Methods,* Amsterdam, Gordon & Breach, 1998.

[2] C. Lubich, *Fractional linear multistep methods for Abel-Volterra integral equations of the second kind,* Math. Comp. 45 (1985) 463-469.

[3] C. Lubich, *A stability analysis of convolution quadrature for Abel-Volterra integral equations,* IMA J. Numer. Anal. 6 (1986) 87-101.

Energetic BEM-FEM for 2D wave propagation problems

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Even if the Finite Element Method has obviously a dominant status in the field of computational techniques in physics and engineering, mostly because of its great flexibility and wide range of applicability, discretization approaches based on integral equations are superior for certain classes of problems. For instance, the numerical study of wave propagation in unbounded media still represents a challenging issue for domain methods. Starting from a recently developed energetic space-time weak formulation for Boundary Integral Equations (BIEs) related to wave propagation problems defined on single and multi domains [1, 2, 3, 4], a coupling algorithm is presented, which allows a flexible use of Finite Element and Boundary Element Methods as local discretization techniques, in order to efficiently treat (unbounded) multilayered media. Partial differential equations associated to BIEs will be weakly reformulated by an energetic approach, too. Taking advantage of the theoretical stability analysis of the coupling technique proposed in [5], where simple one dimensional benchmarks were discussed, here the focus is the extension of the energetic BEM-FEM procedure for the approximate resolution of wave propagation model problems in 2D. Several numerical results will be presented.

References

[1] A. Aimi, M. Diligenti, *A new spacetime energetic formulation for wave propagation analysis in layered media by BEMs,* Int. J. Numer. Meth. Engrg., 75, 1102-1132, (2008).

[2] A. Aimi, S. Gazzola, C. Guardasoni, *Energetic Boundary Element Method analysis of wave propagation in 2D multilayered media,* Math. Methods Appl. Sciences, 35, 1140-1160, (2012).

[3] A. Aimi, M. Diligenti, A. Frangi, C. Guardasoni, *A stable 3D energetic Galerkin BEM approach for wave propagation interior problems,* Engineering Analysis with Boundary Elements, 36, 1756-1765, (2012).

[4] A. Aimi, M. Diligenti, A. Frangi, C. Guardasoni, *Neumann exterior wave propagation problems: computational aspects of 3D energetic Galerkin BEM,* Comp. Mech., 51, 475-493, (2013).

[5] A. Aimi, M. Diligenti, C. Guardasoni, S. Panizzi, *Energetic BEM-FEM coupling for wave propagation in layered media,* Communications in Applied and Industrial Mathematics, 3, No. 2, 1-21, (2012).

Bernstein-Durrmeyer operators with arbitrary weight functions

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We consider a class of Bernstein-Durrmeyer operators with respect to an arbitrary measure on a multi-dimensional simplex. These operators generalize the well-known Bernstein-Durrmeyer operators with Jacobi weights. A motivation for this generalization comes from learning theory.

In the talk, we discuss convergence of the operators. We show that uniform convergence holds for all continuous functions on the simplex if and only if the underlying measure is strictly positive on the simplex. We describe a more special class of strictly positive measures for that we can give an estimate for the rate of convergence. We discuss convergence on the support of non-strictly positive measures. Finally, we prove convergence in the corresponding weighted Lp-spaces and give an estimate for rates of convergence.

This talk is partially based on joint works with Kurt Jetter (University of Hohenheim, Germany) and Bing-Zheng Li (Zhejiang University, P. R. China).

A Prony-like method for recovering monomial-exponential sums

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In this poster we describe a numerical procedure to solve the following non-linear approximation problem: Let $h(x)$ be a monomial-exponential sum

$$h(x) = \sum_{j=1}^{M} \sum_{s=0}^{n_j-1} c_{js} x^s e^{f_j x},$$

where both M and $\{n_j\}_{j=1}^{M}$ are positive integers and $\{c_{js}\}_{j=1}^{M}{}_{s=0}^{n_j-1}$ and $\{f_j\}_{j=1}^{M}$ are complex or real parameters with $c_{jn_j-1} \neq 0$. Setting $L = n_1 + \cdots + n_M$, recover all the parameters of h given $2N(N \geq L)$ sampled data $h(k)$ for $k = k_0, k_0 + 1, \ldots, k_0 + 2N$ with $k_0 \in \mathbb{N}^+ = \{0, 1, 2, \ldots\}$.

This problem occurs in particular in the identification of relevant scattering parameters arising in the solution of non-linear PDEs equations of integrable type by means of the Inverse Scattering Tranform. The method that we propose consists of the following steps:

1. Identification of the common rank of two square Hankel matrices H_0 and H_1 of order N generated by the $2N$ given data;

2. Computation of the parameters M, $\{n_j\}_{j=1}^{M}$ and $\{f_j\}_{j=1}^{M}$ by solving a generalized eigenvalue problem;

3. Computation of the coefficients $\{c_{js}\}_{j=1}^{M}{}_{s=0}^{n_j-1}$ by means of a linear system characterized by the parameters previously identified.

Numerical results will be given in order to highlight the effictiveness of the method.

Numerical approximation of the Mittag-Leffler function and applications in fractional calculus

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The Mittag-Leffler (ML) function, introduced at the beginning of the last century by the Swedish mathematician Magnus Gösta Mittag-Leffler, is nowadays receiving renewed interest because of its applications in fractional calculus; indeed, the ML function plays for fractional differential equations (FDEs) the same key role as the exponential function does for ordinary differential equations (ODEs) of integer order. Motivated by the recent spread of FDEs in modeling real life dynamics, the numerical solution of FDEs is gaining importance. In particular, interesting results come when extending and adapting the methods for ODEs as, for example, exponential integrators. This generalization involves the evaluation of some generalized ML functions in the form

$$e_{\alpha,\beta}(t;\lambda) = t^{\beta-1}E_{\alpha,\beta}(-t^\alpha\lambda), \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where t is an independent variable, λ is a scalar or a matrix and α and β are fixed parameters. The numerical approximation of ML functions, possibly with matrix arguments, is a challenging task. The classical definition in terms of the series is not useful for practical computation because of its slow convergence. Thus, efficient and reliable methods need to be devised. In this poster we present and discuss some methods based on the integral representation of $e_{\alpha,\beta}(t;\lambda)$ and different approaches are compared. For some methods we present a robust error analysis allowing to select the main parameters of the method with the aim of achieving any prescribed accuracy. Some applications in the solution of FDEs are also shown.

Weighted Fejér Constants and Fekete Sets

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We indicate the connections among the Fekete sets, the zeros of orthogonal polynomials, $1(w)$-normal point systems, and the nodes of an interpolatory process which is called stable and the most economical, via the Fejér constants. As an application of the results, a convergence theorem on Grünwald interpolatory process on the real line for Freud-type weights is given.

Approximation processes related to pseudo-differential operators

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Let $q : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{C}$ be a negative definite symbol, i.e., q is measurable and locally bounded in (x, ξ) and continuous and negative definite as a function of ξ.

The pseudo-differential operator A associated with q has the following Lévy-Khincin representation on $C^2(\mathbb{R}^N)$

$$Au(x) = -q(x, D) = \sum_{i,j=1}^{N} a_{ij}(x) D_{ij}^2 u(x) + b(x) \cdot \nabla u(x) + c(x) u(x)$$

$$+ \int_{\mathbb{R}^N \setminus \{0\}} \left(u(x+y) - u(x) - \frac{y \cdot \nabla u(x)}{1 + |y|^2} \right) d\nu(x, \cdot),$$

where, for every $x \in \mathbb{R}^N$, $(a_{ij}(x))_{ij}$ is a positive semidefinite matrix, $b(x) \in \mathbb{R}^N$, $c(x) \leq 0$ and ν is a Lévy kernel on $\mathbb{R}^N \times B(\mathbb{R}^N \setminus \{0\})$.

For every $x \in \mathbb{R}^N$ and $n \geq 1$, denoted by $\mu_{x,n}$ the probability measure on $\mathbb{R}^N$ having characteristic function $e^{ix\xi - \frac{q(x,\xi)}{n}}$, the positive linear operator

$$L_n u(x) = \int_{\mathbb{R}^N} u(y) d\mu_{x,n}(y)$$

makes sense on bounded Borel-measurable functions on $\mathbb{R}^N$.

We prove that, under suitable regularity assumptions on the symbol q, such as the continuity with respect to x and the differentiability with respect to ξ, these operators are well defined on the Banach space $C_0^w(\mathbb{R}^N)$ of the function $u \in C(\mathbb{R}^N)$ such that wu vanishes at infinity.

We also demonstrate that the sequence $(L_n)_n$ is an approximation process on $C_0^w(\mathbb{R}^N)$ and the Voronovskaja-type formula

$$\lim_n n(L_n u - u) = -q(x, D) u$$

holds on $K^2(\mathbb{R}^N)$.

Moreover, when $-q(x, D)$ has an extension generating a positive C_0-semigroup $(T(t))_{t \geq 0}$ on $C_0^w(\mathbb{R}^N)$ satisfying the Feller property with respect to $C_0(\mathbb{R}^N)$ (see [1]), we get, for every $t \geq 0$, the approximation by iterates

$$T(t)u = \lim_n L_n^{k_n} u$$

for every $u \in C_0^w(\mathbb{R}^N)$, $(k_n)_n$ being a sequence of positive integers such that $k_n/n \to t$.

Our results improve those of [2], where the coefficients of the pseudo-differential operator $-q(x, D)$ are bounded and the setting is $C_0(\mathbb{R}^N)$.

References

[1] F. Altomare, S. Milella and G. Musceo, , *On a class of positive C_0- semi-groups of operators on weighted continuous function spaces,* Note Mat., vol. 31, 1 (2011), 15-28.

[2] B. Bottcher and R. Schilling , *Approximation of Feller processes by Markov chains with Lévy increments,* Stochastic and Dynamics, vol.9, no.1 (2009), 71-80.

Generalized Stieltjes Functions

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A function f defined on $(0, \infty)$ is called a generalized Stieltjes function of order λ $(\lambda > 0)$ if it can be written in the form

$$f(x) = c + \int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda},$$

where $c \geq 0$, and μ is a positive measure on $[0, \infty)$ making the integral converge for all $x > 0$.

A function g defined on $(0, \infty)$ is called completely monotonic of order l if $x^l g(x)$ is completely monotonic, i.e. if $(-1)^k (x^l g(x))^{(k)} > 0$ for all $x > 0$ and $k \geq 0$. The class of completely monotonic functions of order l has been investigated and characterized in terms of Laplace transforms of positive measures (similar to Bernstein's theorem, relating completely monotonic functions and Laplace transforms).

In this presentation we shall characterize the generalized Stieltjes functions f corresponding to measures having moments of all orders in terms of complete monotonicity of positive order of the remainders in asymptotic expansions of the functions $x^{\lambda-1} f(x)$.

In the case $\lambda = 1$ the characterization is in fact a half-line analogue of a result from around 1922 of Hamburger and Nevanlinna relating asymptotic expansions to the classical Hamburger moment problem.